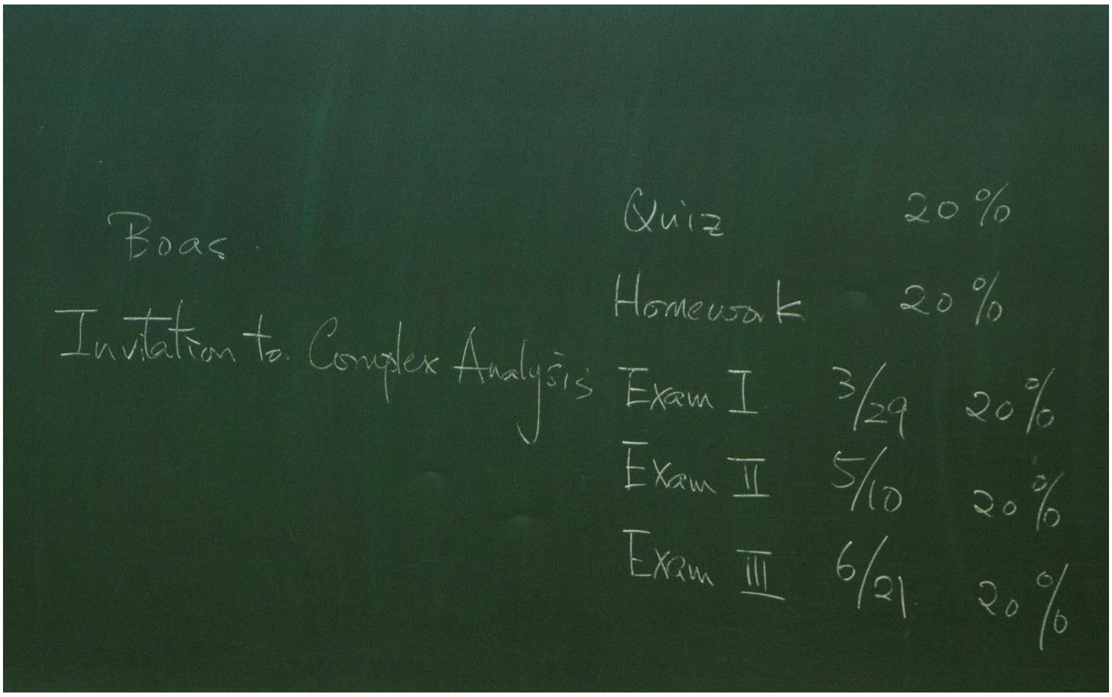
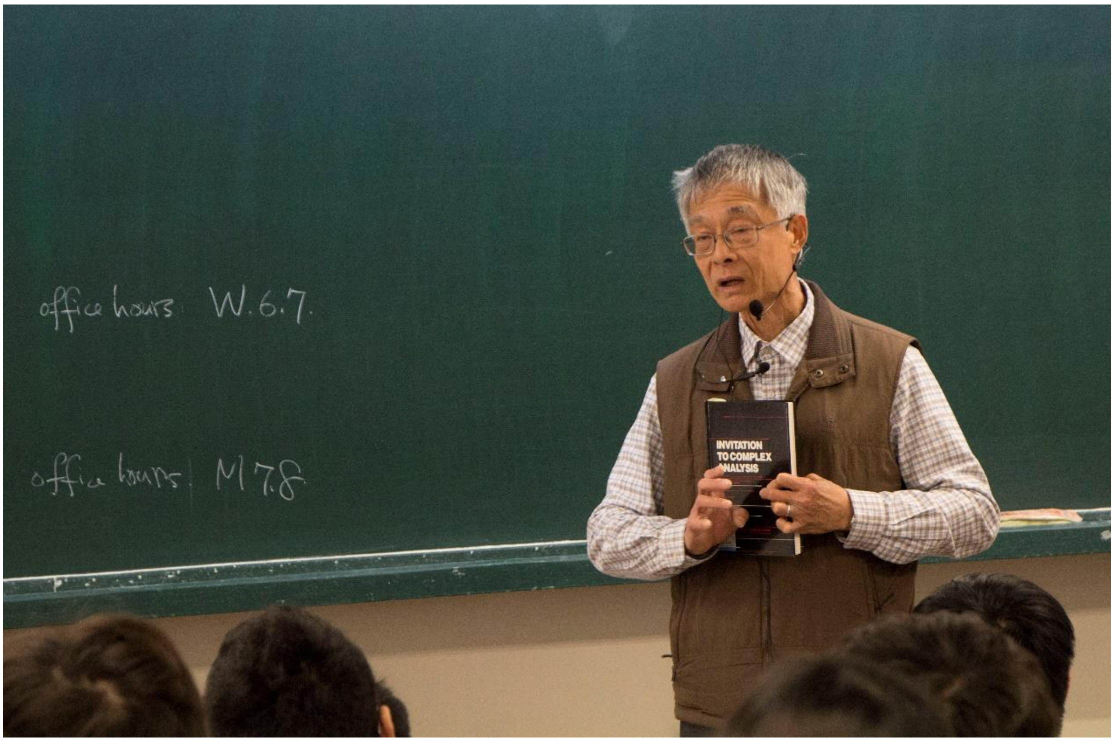
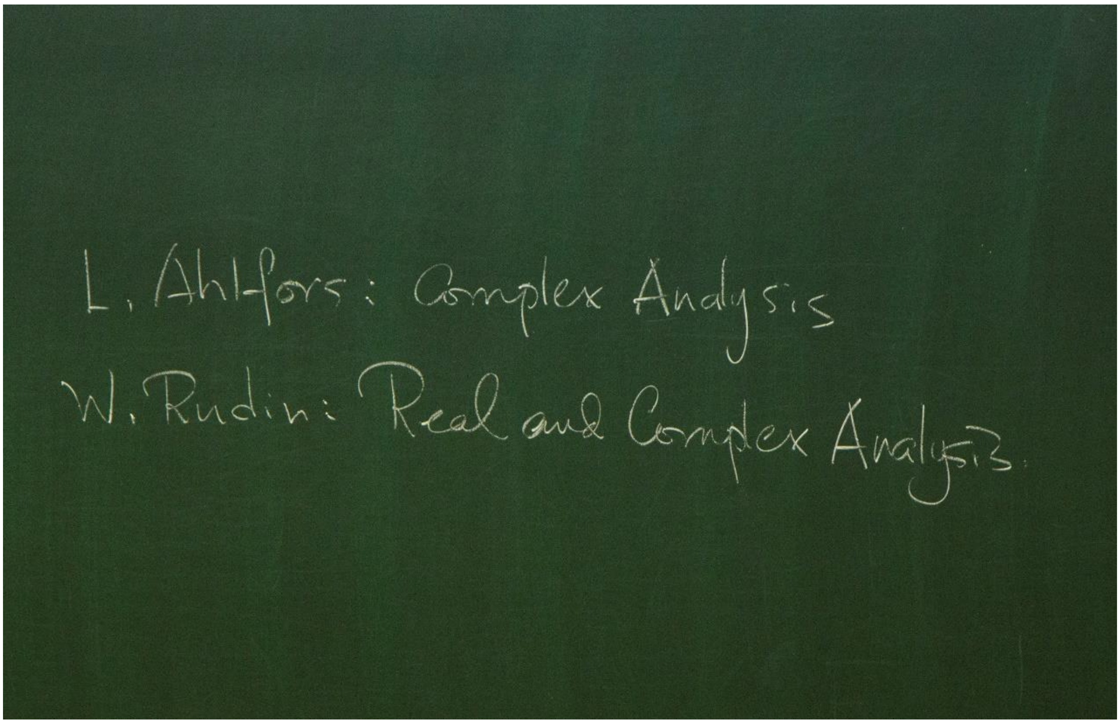
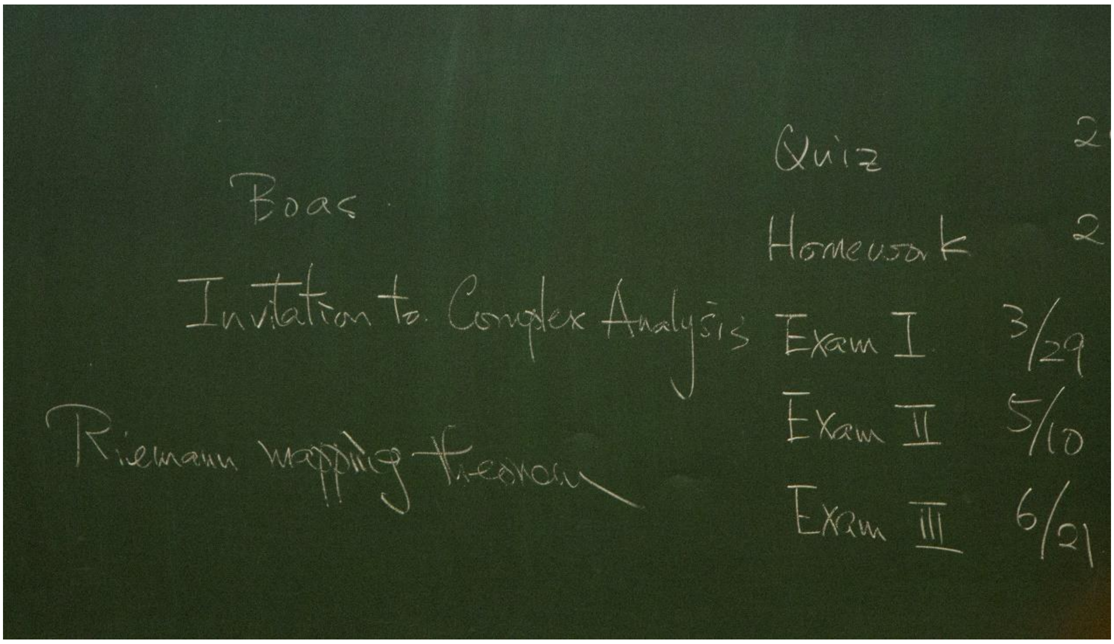
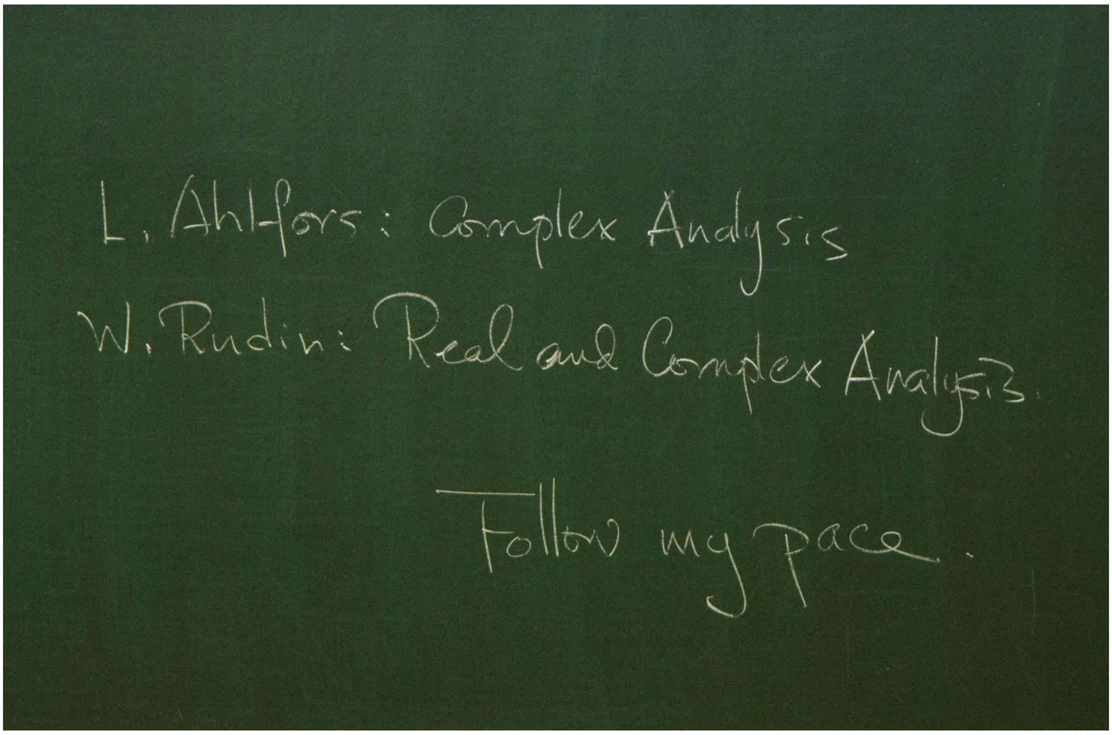
【10920 程守慶教授複變數函數論 / 第 1 堂版書】

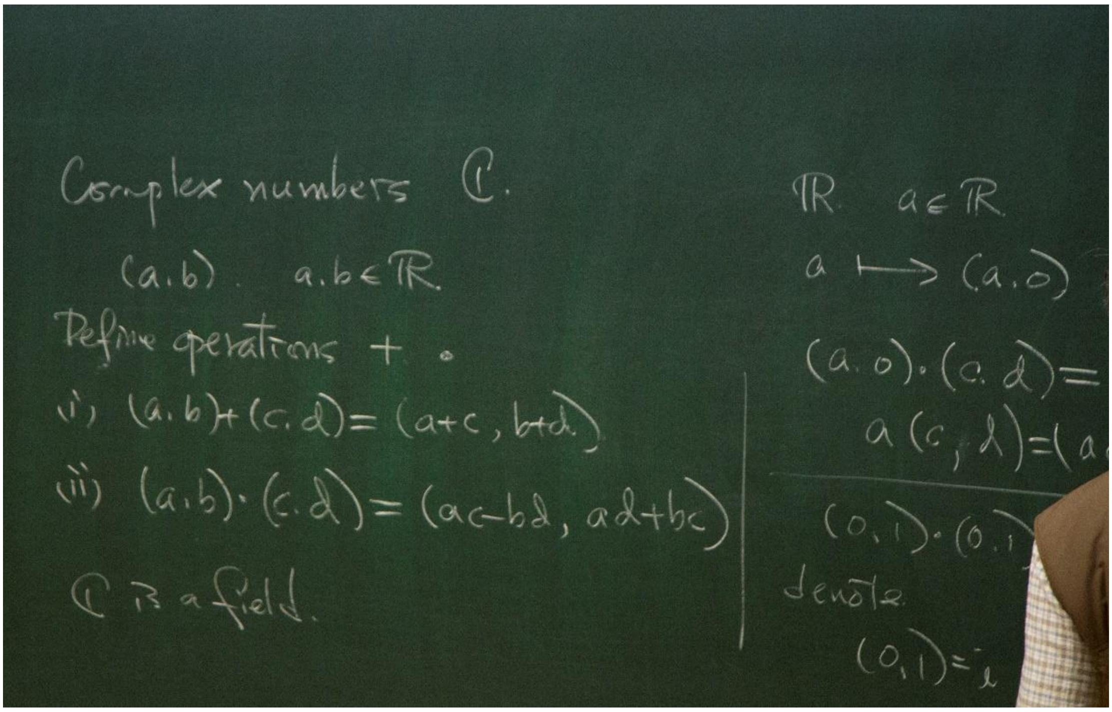





L. Ahlfors: Complex Analysis
W. Rudin: Real and Complex Analysis.


Boas.
Invitation to Complex Analysis
Riemann mapping theorem
Quiz 2
Homework 2
Exam I 3/29
Exam II 5/10
Exam III 6/21


L. Ahlfors: Complex Analysis
W. Rudin: Real and Complex Analysis.
Follow my pace.


Complex numbers $\mathbb{C}$.
$(a.b)$. $a.b \in \mathbb{R}$.
Define operations $+$ $\cdot$
i) $(a.b)+(c.d)=(a+c, b+d)$
ii) $(a.b)\cdot(c.d)=(ac-bd, ad+bc)$
$\mathbb{C}$ is a field.
$\mathbb{R}$. $a \in \mathbb{R}$
$a \longmapsto (a.0)$
$(a.0)\cdot(c.d)=$
$a(c,d)=(a$
$(0,1)\cdot(0,1$
denote
$(0,1)=i$

$\mathbb{R}$. $\quad a \in \mathbb{R}$

$a \longmapsto (a,0)$

$(a,0)\cdot(c,d) = (ac, ad)$

$a(c,d) = (ac, ad)$

$(0,1)\cdot(0,1) = (0-1, 0) = (-1,0)$

denote

$(0,1) = i \quad , \quad i^2 = -1 \qquad x^2+1=0$

$z \in \mathbb{C}$

$z = a+ib$.

$z \in \mathbb{C}$

$z = a+ib$.

$$
\begin{aligned}
z &= (a,b) \\
&= (a,0)+(0,b) \\
&= a + b(0,1) \\
&= a+ib.
\end{aligned}
$$

$a = \mathrm{Re}\, z$ $\qquad$ $\mathrm{Im}\, z = b$

实部 $\qquad$ 虛部

Real part $\qquad$ imaginary part

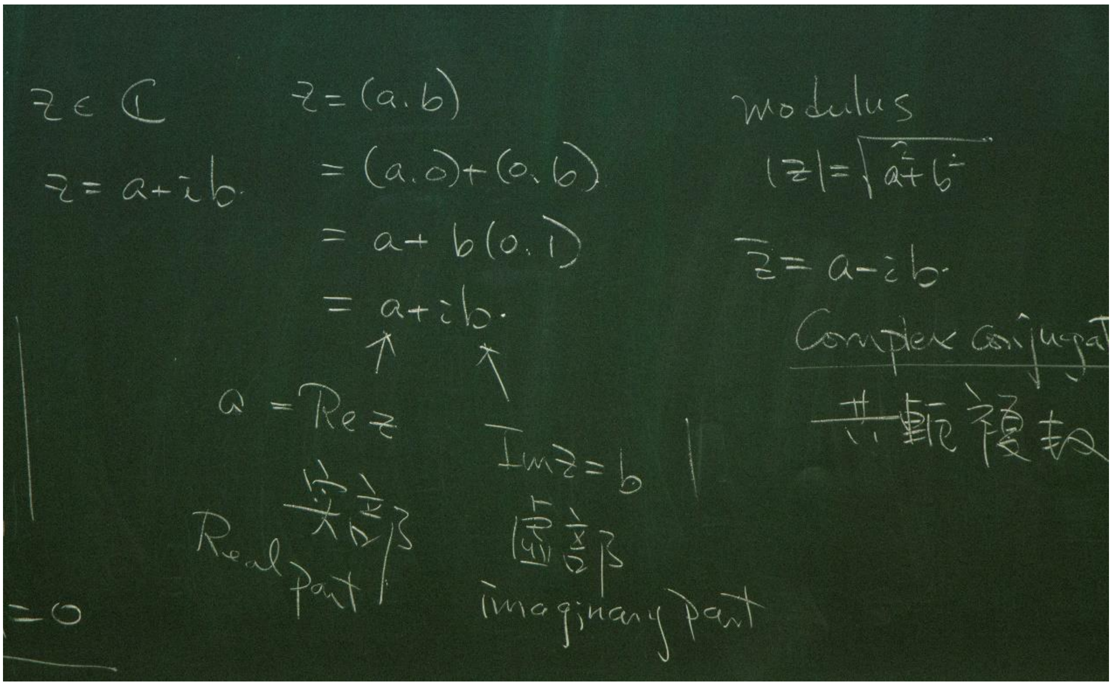
modulus

$|z| = \sqrt{a^2+b^2}$

$\bar{z} = a - ib$.

Complex conjugate

共軛複數



$z \in \mathbb{C}$
$z = a + ib.$
$z = (a,b)$
$= (a,0) + (0,b)$
$= a + b(0,1)$
$= a + ib.$
$a = \mathrm{Re}\, z$
實部
Real part
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imaginary part
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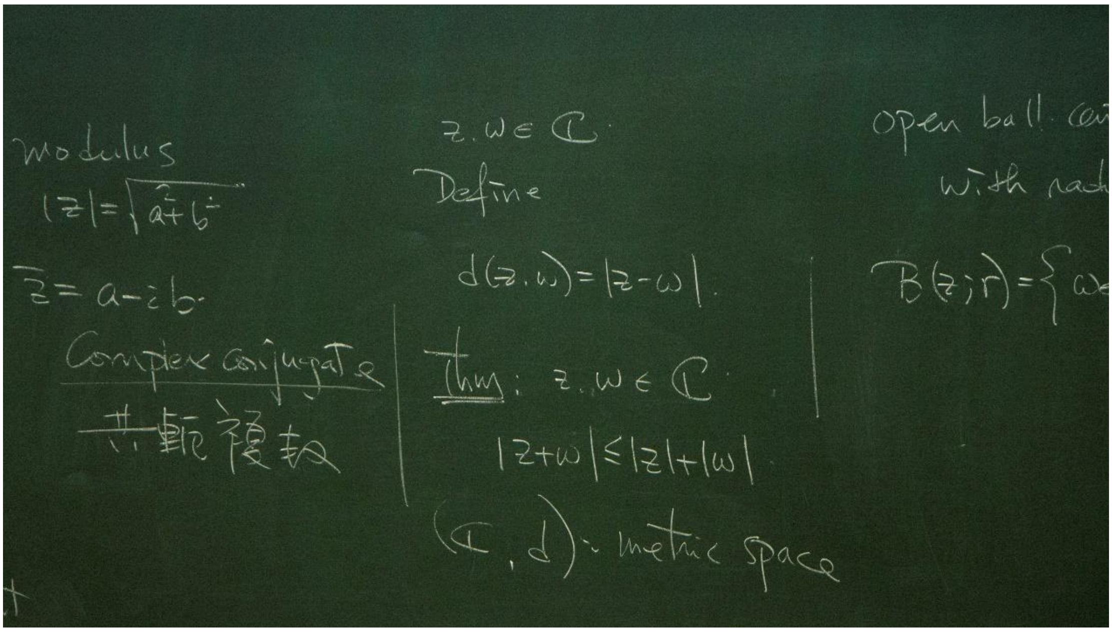

modulus
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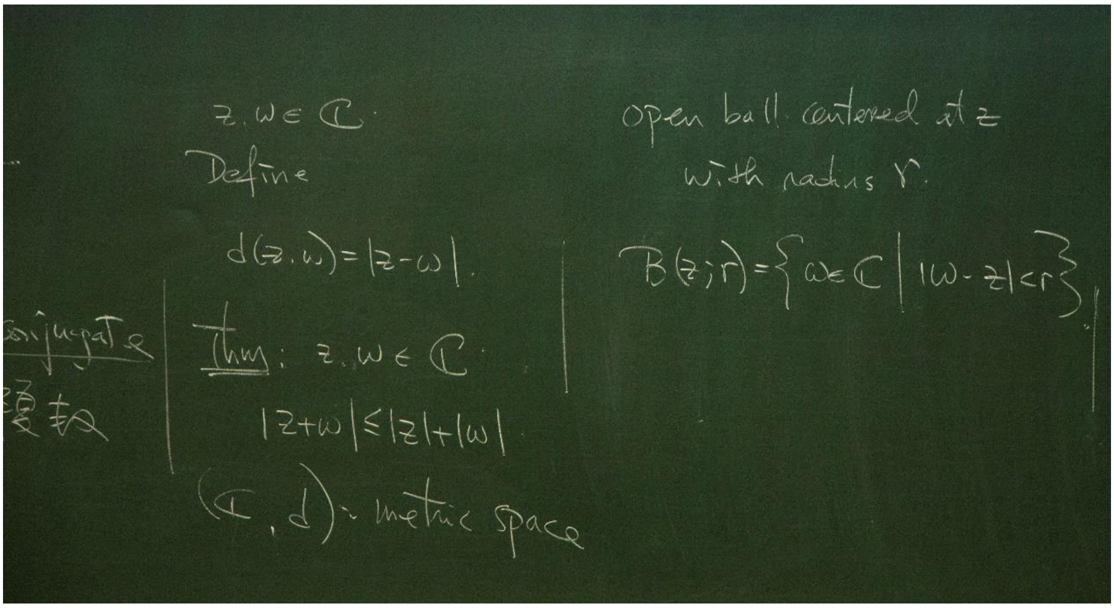
Complex conjugate
共軛複數
$z, w \in \mathbb{C}.$
Define
$d(z,w) = |z - w|.$
Thm: $z, w \in \mathbb{C}$
$|z + w| \le |z| + |w|.$
$(\mathbb{C}, d)$: metric space
open ball.
$B(z;r) = \{$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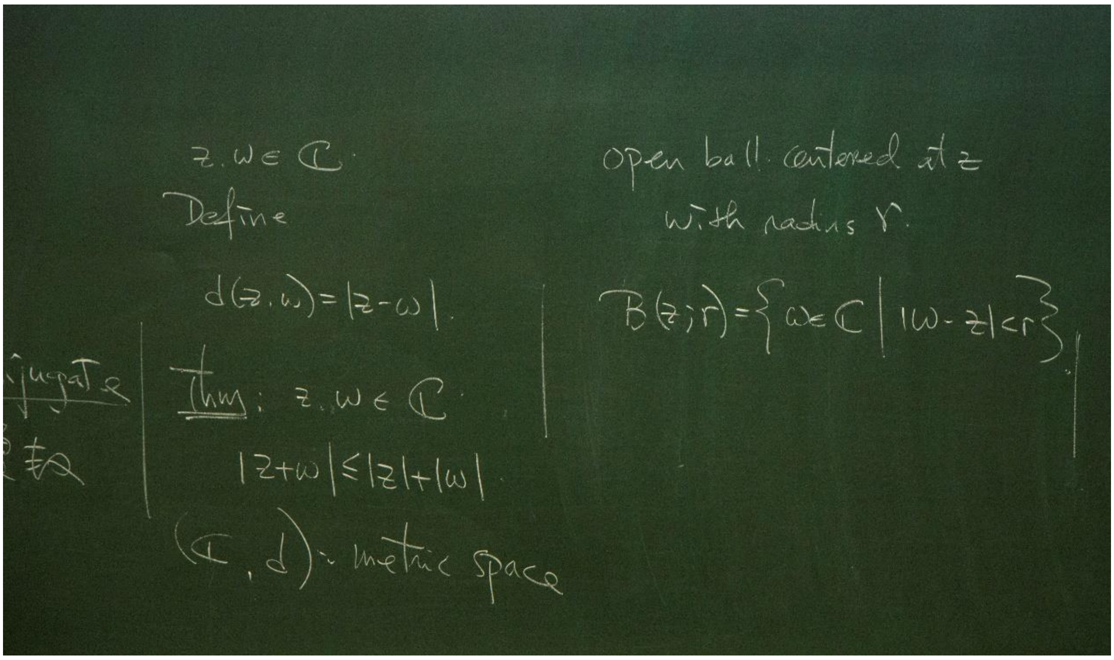
複數
Thm: $z, w \in \mathbb{C}$
$|z+w| \le |z| + |w|$
$(\mathbb{C}, d)$: metric space
open ball centered at z
with radius r.
$B(z;r) = \{ w \in \mathbb{C} \mid |w-z| < r \}$.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$f : \Omega \longrightarrow \mathbb{C}$ $\quad \Omega \subseteq \mathbb{C}$ domain

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

$z_0 \in \Omega$. $\quad$ Connected open

f is said to be differentiable at z_0.

if $\displaystyle\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.

denoted by $f'(z_0)$

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If f is diff. at every point of Ω

then we say f is a holomorphic function on Ω.

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$S \subseteq \mathbb{C}$ set.

$f \in \mathcal{O}(S)$ means. f is holomorphic on some open neighborhood of S.

If f is diff. at z_0,

$z \longrightarrow z_0$. along x-axis.

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0+h, y_0)+iv(x_0+h, y_0)-(u(x_0, y_0)+iv(x_0, y_0))}{h}$$

$$= \frac{\partial u}{\partial x}(z_0)+i\frac{\partial v}{\partial x}(z_0)$$

$z \to z_0$ along y-axis

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{k\to 0} \frac{u(x_0, y_0+k) + i v(x_0, y_0+k) - (u(x_0,y_0) + i v(x_0,y_0))}{ik}$$

$$= \frac{1}{i}\left(\frac{\partial u}{\partial y}(z_0) + i\frac{\partial v}{\partial y}(z_0)\right)$$

$$= \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0)$$

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$$\frac{\partial u}{\partial x}(z_0)=\frac{\partial v}{\partial y}(z_0)$$

Cauchy-Riemann eq.

$$\frac{\partial u}{\partial y}(z_0)=-\frac{\partial v}{\partial x}(z_0)$$

柯西-黎曼方程式

$$\frac{\partial u}{\partial x}(0,0)=1,\quad \frac{\partial u}{\partial y}(0,0)=-1$$

$$\frac{\partial v}{\partial x}(0,0)=1\quad \frac{\partial v}{\partial y}(0,0)=1.$$

$$\frac{\partial u}{\partial x}(z_0)=\frac{\partial v}{\partial y}(z_0)$$

Cauchy-Rie

$$\frac{\partial u}{\partial y}(z_0)=-\frac{\partial v}{\partial x}(z_0)$$

柯西-黎

along x-axis

$$\frac{f(x,0)-f(0,0)}{x}=\frac{x+ix-0}{x}=1+i$$

along x=y

$$\frac{f(x,x)-f(0,0)}{x+ix}=\frac{ix}{x+ix}=\frac{i}{1+i}=\frac{i(1-i)}{(1+i)(1-i)}=\frac{1+i}{2}$$

$\frac{\partial u}{\partial x}(0,0)=1, \quad \frac{\partial u}{\partial y}(0,0)=-1$

$\frac{\partial v}{\partial x}(0,0)=1. \quad \frac{\partial v}{\partial y}(0,0)=1.$

$\frac{\partial u}{\partial x}(z_0)=\frac{\partial v}{\partial y}(z_0)$ Cauchy-Rieman

$\frac{\partial u}{\partial y}(z_0)=-\frac{\partial v}{\partial x}(z_0)$ 柯西-黎曼

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$$\frac{f(x,x)-f(0,0)}{x+ix}=\frac{ix}{x+ix}=\frac{i}{1+i}=\frac{i(1-i)}{(1+i)(1-i)}=\frac{1+i}{2}$$

If $f\in C^1(\Omega)$ Set $\frac{\partial}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$ Cauchy-Riemann operator

$$\frac{\partial f}{\partial \bar z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(u+iv)$$
$$=\frac{1}{2}\left(\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)$$
$$=0$$

$\Longleftrightarrow f\in\mathcal{O}(\Omega)$

$$\boxed{\frac{\partial u}{\partial \bar z}=g}$$

Curve:

$\gamma:[a,b]\longrightarrow\mathbb{C}$

$t\longmapsto\gamma(t)=(x(t),y(t)).$

Smooth curve

$\gamma\in C^1([a,b])$ and $\gamma'(t)\neq 0$ on (a,b)

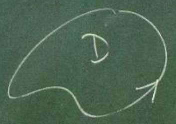
Curve: piecewise smooth

$\gamma:[a,b]\longrightarrow\mathbb{C}$

$\exists\ P=\{a=t_0<t_1<\cdots<t_{n-1}<t_n=b\}$

s.t. γ is smooth on each $[t_{j-1},t_j]$, $1\le j\le n$.

Thm (Green)



D is a bounded domain in $\mathbb{R}^2$ with piecewise smooth boundary ∂D. Let P, Q be two functions which is C^1 in some open neighborhood of $\overline{D}$. Assume ∂D is positively oriented.

Then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Thm. Let D be a simply connected domain in $\mathbb{C}$. Suppose $f \in C^1(D)$. Then

$$f \in \mathcal{O}(D) \iff \oint_\Gamma f(z)\,dz = 0$$

for every piecewise smooth simple closed curve Γ in D.

arcwise connected + $\pi_1(D)=0$

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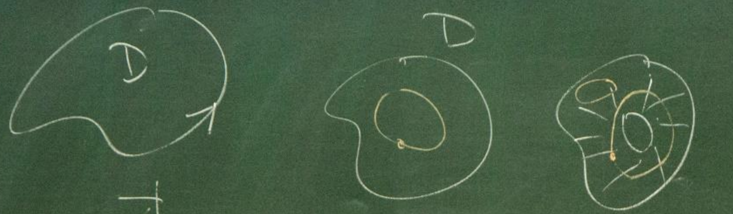

Then

∂D.

B.

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

thm. Let D b

Suppose $f \in$

$f \in \mathcal{O}(D)$

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Thm. Let D be a simply connected domain in $\mathbb{C}$.

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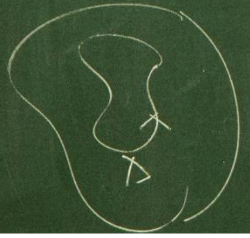

D

$dz = dx + i\,dy$

arcwise connected + $\pi_1(D)=0$

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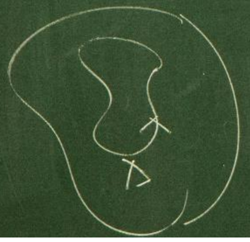

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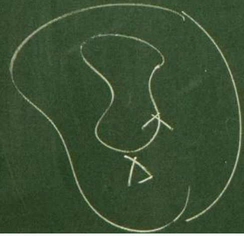

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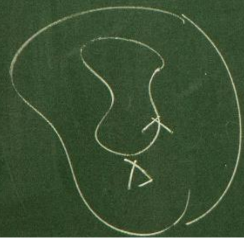


D

$dz = dx + i\,dy$.

$$\oint_P f(z)\,dz = \oint_P f(z)\,(dx+i\,dy) = \iint_\Omega \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy.$$

$$= i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

Ω: region surrounded by P.

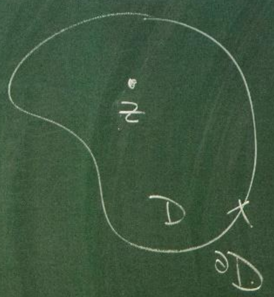
$$= 2i \iint_\Omega \frac{\partial f}{\partial \bar z}\, dx\,dy.$$

<u>thm</u>: (Cauchy's integral formula)

Let D be a bounded domain in $\mathbb{C}$ with piecewise smooth boundary.

Suppose $f \in C^1(\bar D)$. Then

$w = x+iy$



$$f(z) = \frac{1}{2\pi i}\oint_{\partial D} \frac{f(w)}{w-z}\,dw - \frac{1}{\pi}\iint_D \frac{\frac{\partial f}{\partial \bar w}(w)}{w-z}\,dx\,dy, \quad z \in D.$$